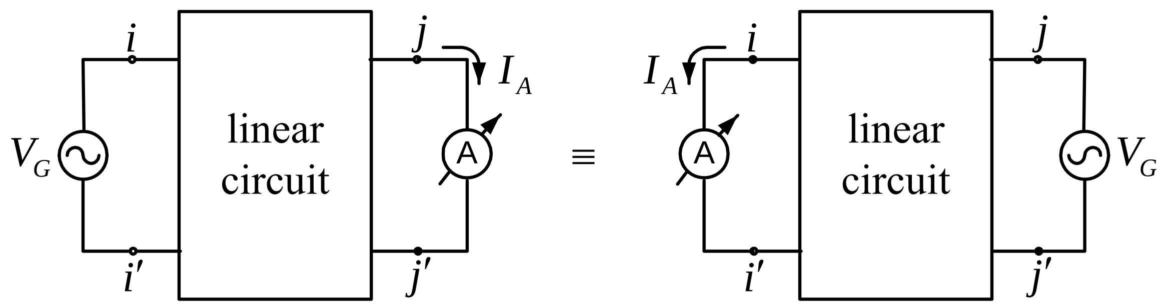
LECTURE 10: Reciprocity. Cylindrical Antennas – Analytical Models

(*Reciprocity theorem. Implications of reciprocity in antenna measurements. Self-impedance of a dipole using the induced* ***emf*** *method. Pocklington's equation. Hallén's equation.*)

1. Reciprocity theorem for antennas

1.1. Reciprocity theorem in circuit theory

If a voltage (current) generator is placed between any pair of nodes of a linear circuit, and a current (voltage) reaction is measured between any other pair of nodes, the interchange of the generator's and the measurement's locations would lead to the same measurement results.



$$\frac{V_i}{I_j} = \frac{V_j}{I_i} \quad \text{or} \quad Z_{ji} = Z_{ij}\,. \tag{10.1}$$

1.2. Reciprocity theorem in EM field theory (Lorentz' reciprocity theorem)

Consider a volume $V_{[S]}$ bounded by the surface S, where two pairs of sources exist: $(\mathbf{J}_1, \mathbf{M}_1)$ and $(\mathbf{J}_2, \mathbf{M}_2)$. We denote the field associated with the $(\mathbf{J}_1, \mathbf{M}_1)$ sources as $(\mathbf{E}_1, \mathbf{H}_1)$, and the field generated by $(\mathbf{J}_2, \mathbf{M}_2)$ as $(\mathbf{E}_2, \mathbf{H}_2)$.

$$\left| \begin{array}{ll} \nabla \times \mathbf{E}_1 = -j\omega\mu \mathbf{H}_1 - \mathbf{M}_1 & \quad / \cdot \mathbf{H}_2 \\ \nabla \times \mathbf{H}_1 = j\omega\varepsilon \mathbf{E}_1 + \mathbf{J}_1 & \quad / \cdot \mathbf{E}_2 \end{array} \right. \tag{10.2}$$

$$\left| \begin{array}{ll} \nabla \times \mathbf{E}_2 = -j\omega\mu \mathbf{H}_2 - \mathbf{M}_2 & \quad / \cdot \mathbf{H}_1 \\ \nabla \times \mathbf{H}_2 = j\omega\varepsilon \mathbf{E}_2 + \mathbf{J}_2 & \quad / \cdot \mathbf{E}_1 \end{array} \right. \tag{10.3}$$

The vector identity

$$\nabla\cdot(\mathbf{E}_1\times\mathbf{H}_2-\mathbf{E}_2\times\mathbf{H}_1)=\mathbf{H}_2\cdot\nabla\times\mathbf{E}_1-\mathbf{E}_1\cdot\nabla\times\mathbf{H}_2-\mathbf{H}_1\cdot\nabla\times\mathbf{E}_2+\mathbf{E}_2\cdot\nabla\times\mathbf{H}_1$$

is used to obtain

$$\nabla\cdot(\mathbf{E}_1\times\mathbf{H}_2-\mathbf{E}_2\times\mathbf{H}_1)=-\mathbf{E}_1\cdot\mathbf{J}_2+\mathbf{H}_1\cdot\mathbf{M}_2+\mathbf{E}_2\cdot\mathbf{J}_1-\mathbf{H}_2\cdot\mathbf{M}_1. \qquad (10.4)$$

Equation (10.4) is written in its integral form as

$$\oiint_S(\mathbf{E}_1\times\mathbf{H}_2-\mathbf{E}_2\times\mathbf{H}_1)\cdot d\mathbf{s}=\iiint_{V[S]}(-\mathbf{E}_1\cdot\mathbf{J}_2+\mathbf{H}_1\cdot\mathbf{M}_2+\mathbf{E}_2\cdot\mathbf{J}_1-\mathbf{H}_2\cdot\mathbf{M}_1)dv. \qquad (10.5)$$

Equations (10.4) and (10.5) represent the general Lorentz reciprocity theorem in differential and integral forms, respectively.

One special case of the reciprocity theorem is of fundamental importance to antenna theory, namely, its application to unbounded (open) problems. In this case, the surface S is a sphere of infinite radius. Therefore, the fields integrated over it are far-zone fields. This means that the left-hand side of (10.5) vanishes:

$$\oiint_S\left(\frac{|\mathbf{E}_1||\mathbf{E}_2|}{\eta}\cos\gamma-\frac{|\mathbf{E}_1||\mathbf{E}_2|}{\eta}\cos\gamma\right)ds=0. \qquad (10.6)$$

Here, γ is the angle between the polarization vectors of both fields, $\mathbf{E}_1$ and $\mathbf{E}_2$. Note that in the far zone, the field vectors are orthogonal to the direction of propagation and, therefore are orthogonal to $d\mathbf{s}$. Thus, in the case of open problems, the reciprocity theorem reduces to

$$\iiint_{V[S]}(\mathbf{E}_1\cdot\mathbf{J}_2-\mathbf{H}_1\cdot\mathbf{M}_2)dv=\iiint_{V[S]}(\mathbf{E}_2\cdot\mathbf{J}_1-\mathbf{H}_2\cdot\mathbf{M}_1)dv. \qquad (10.7)$$

Each of the integrals in (10.7) can be interpreted as ***coupling energy*** between the field produced by some sources and another set of sources generating another field. The quantity

$$\langle 1,2\rangle=\iiint_{V[S]}(\mathbf{E}_1\cdot\mathbf{J}_2-\mathbf{H}_1\cdot\mathbf{M}_2)dv$$

is called the ***reaction*** of the field $(\mathbf{E}_1,\mathbf{H}_1)$ to the sources $(\mathbf{J}_2,\mathbf{M}_2)$. Similarly,

$$\langle 2,1\rangle=\iiint_{V[S]}(\mathbf{E}_2\cdot\mathbf{J}_1-\mathbf{H}_2\cdot\mathbf{M}_1)dv$$

is the ***reaction*** of the field $(\mathbf{E}_2,\mathbf{H}_2)$ to the sources $(\mathbf{J}_1,\mathbf{M}_1)$. Thus, in shorthand, the reciprocity equation (10.7) is $\langle 1,2\rangle=\langle 2,1\rangle$.

The Lorentz reciprocity theorem is the most general form of reciprocity in linear electromagnetic systems. Circuit theory reciprocity is a special case of lumped element sources and responses (local voltage or current measurements).

To illustrate the above statement, consider the following scenario. Assume that the sources in two measurements have identical amplitude-phase distributions in their respective volumes. Note that the volumes may reside in different locations with respect to a global coordinate system. However, with each source volume (#1 or #2), we can associate a local coordinate system where the position is given by $\mathbf{x}_{1,2} = (r_{1,2}, \theta_{1,2}, \varphi_{1,2})$. If the sources have identical distributions, then, the source volumes are the same in shape and size, $V_1 = V_2 = V_s$, and $\mathbf{J}_1(\mathbf{x}_1) = \mathbf{J}_2(\mathbf{x}_2) = \mathbf{J}$, and $\mathbf{M}_1(\mathbf{x}_1) = \mathbf{M}_2(\mathbf{x}_2) = \mathbf{M}$. According to (10.7),

$$\iiint\limits_{V_s} (\mathbf{E}_1 \cdot \mathbf{J} - \mathbf{H}_1 \cdot \mathbf{M}) dv_2 = \iiint\limits_{V_s} (\mathbf{E}_2 \cdot \mathbf{J} - \mathbf{H}_2 \cdot \mathbf{M}) dv_1 . \tag{10.8}$$

It follows that $\mathbf{E}_1 = \mathbf{E}_2$ and $\mathbf{H}_1 = \mathbf{H}_2$. Note that the above result is general and holds for any mutual position of the two source sets. Also, nothing will change if we now position the sources of set #1 in V_2 and the sources of set #2 in V_1. ***The reaction (measured field) is insensitive to the interchange of source and measurement locations.*** This is essentially the same principle that is postulated as reciprocity in circuit theory (see Section 1.1). Only that we consider volumes instead of nodes and branches, and field vectors instead of voltages and currents.

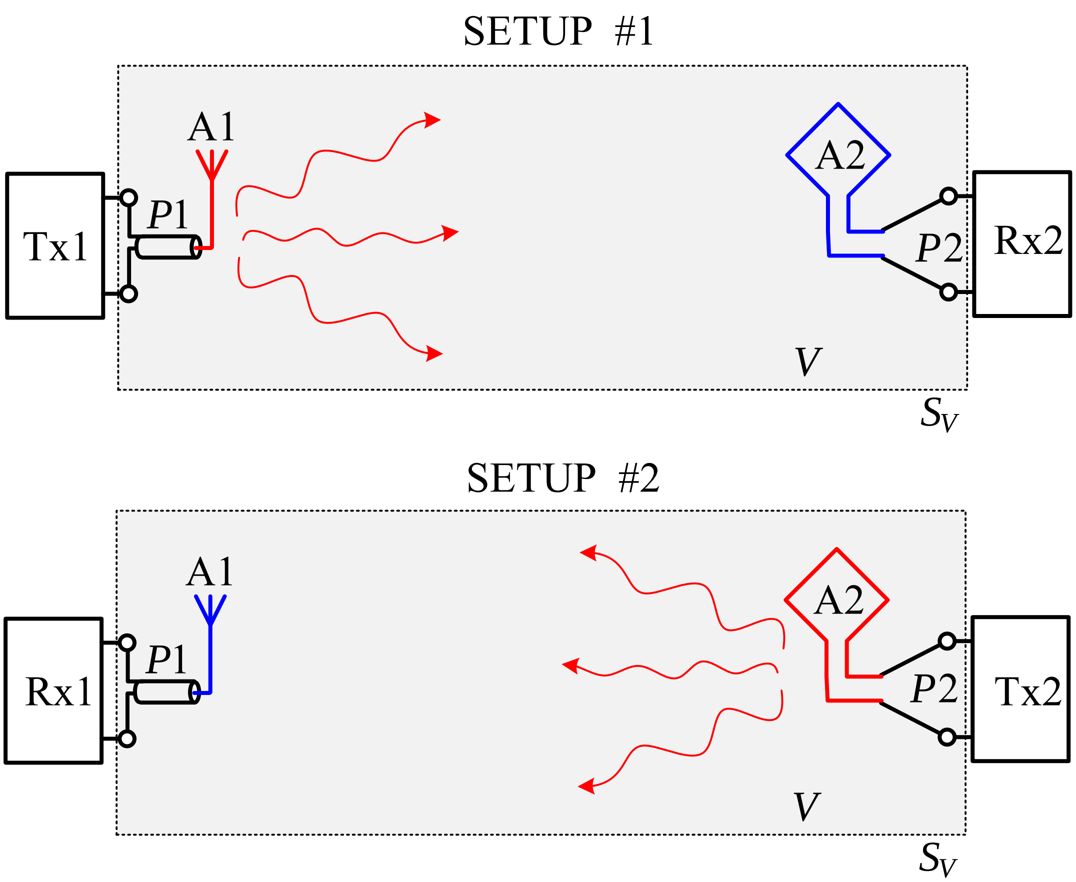
The general reciprocity theorem can be postulated also as: *any network constructed of linear isotropic matter has a symmetrical impedance matrix*. This "network" can be two antennas and the space between them.

1.3. Implications of reciprocity for the received-to-transmitted power ratio

Using the reciprocity theorem, we next prove that ***the ratio of received to transmitted power P_r / P_t does not depend on whether antenna #1 transmits and antenna #2 receives or vice versa***. We should reiterate that the reciprocity theorem holds only if the whole system (antennas + propagation environment) is isotropic and linear.

In this case, we view the two antenna system as a two-port microwave network; see the figure below. Port 1 (P1) connects to antenna 1 (A1) while port 2 (P2) is at the terminals of antenna 2 (A2). Depending on whether an

antenna transmits or receives, its terminals are connected to a transmitter (Tx) or a receiver (Rx), respectively. We consider two measurement setups. In Setup #1, A1 transmits and A2 receives while in Setup #2 A1 receives and A2 transmits.



The volume V in both setups excludes the power sources in the respective transmitters and, therefore, it does not have impressed currents sources, i.e., $\mathbf{J}_1 = \mathbf{J}_2 = 0$ and $\mathbf{M}_1 = \mathbf{M}_2 = 0$. The reciprocity in integral (10.5) becomes

$$\oiint_{S_V} (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) \cdot d\mathbf{s} = 0 \qquad (10.9)$$

where the subscripts refer to the measurement setups. The surface S_V extends to infinity away from the antennas but it also crosses through P1 and P2. At infinity, the integration in (10.9) has no contribution; however, at the cross-sections S_1 and S_2 of ports 1 and 2, respectively, the contributions are not zero. Then,

$$\iint_{S_1} (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) \cdot d\mathbf{s} + \iint_{S_2} (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) \cdot d\mathbf{s} = 0 . \qquad (10.10)$$

Let us now assume that the transmit power in both setups is set to 1 W. Let us denote the field vectors in the transmission lines of ports 1 and 2 corresponding to 1-W transferred power as $(\mathbf{e}_{P1}, \mathbf{h}_{P1})$ and $(\mathbf{e}_{P2}, \mathbf{h}_{P2})$, respectively.[1] We assume that these vectors correspond to power transfer *from* the antenna (outward propagation). When the power is transferred toward the antenna, due to the opposite direction of propagation, we have to change the sign of either the **e** or the **h** vector (but not both!) in the respective pair.

At P1, in Setup #1, the incident field is the 1-W field generated by Tx1, which is $(\mathbf{e}_{P1}, -\mathbf{h}_{P1})$. There could be a reflected field due to impedance mismatch at the A1 terminals, which can be expressed as $\Gamma_1(\mathbf{e}_{P1}, \mathbf{h}_{P1})$ where Γ_1 is the reflection at P1. Also, at P1, in Setup #2, there is the field $(\mathbf{E}_2,\mathbf{H}_2)$ due to the radiation from A2. Analogous field components can be identified at P2 in both setups. Equation (10.10) now becomes

$$\begin{aligned}&\iint_{S_1}\left[(\mathbf{e}_{P1}\times\mathbf{H}_2+\mathbf{E}_2\times\mathbf{h}_{P1})+\Gamma_1(\mathbf{e}_{P1}\times\mathbf{H}_2-\mathbf{E}_2\times\mathbf{h}_{P1})\right]\cdot d\mathbf{s}+\\&\iint_{S_2}\left[(-\mathbf{E}_1\times\mathbf{h}_{P2}-\mathbf{e}_{P2}\times\mathbf{H}_1)+\Gamma_2(\mathbf{E}_1\times\mathbf{h}_{P2}-\mathbf{e}_{P2}\times\mathbf{H}_1)\right]\cdot d\mathbf{s}=0.\end{aligned}\tag{10.11}$$

Next, the received fields in both scenarios, $(\mathbf{E}_1,\mathbf{H}_1)$ at P2 and $(\mathbf{E}_2,\mathbf{H}_2)$ at P1, can be expressed in terms of $(\mathbf{e}_{P2}, \mathbf{h}_{P2})$ and $(\mathbf{e}_{P1}, \mathbf{h}_{P1})$, which represent 1-W received powers at the respective ports:

$$\begin{aligned}(\mathbf{E}_2,\mathbf{H}_2)\big|_{P1}&=R_{1,2}(\mathbf{e}_{P1},\mathbf{h}_{P1})\\(\mathbf{E}_1,\mathbf{H}_1)\big|_{P2}&=R_{2,1}(\mathbf{e}_{P2},\mathbf{h}_{P2}).\end{aligned}\tag{10.12}$$

Note that (10.12) implies that the respective received-to-transmitted power ratios in Setup #1 and Setup #2 are

$$P_{r1}/P_{t1}=R_{2,1}^2 \tag{10.13}$$

$$P_{r2}/P_{t2}=R_{1,2}^2. \tag{10.14}$$

Substituting (10.12) into (10.11) leads to

$$\begin{aligned}&R_{1,2}\iint_{S_1}\Big[\underbrace{(\mathbf{e}_{P1}\times\mathbf{h}_{P1}+\mathbf{e}_{P1}\times\mathbf{h}_{P1})}_{=4}+\Gamma_1\underbrace{(\mathbf{e}_{P1}\times\mathbf{h}_{P1}-\mathbf{e}_{P1}\times\mathbf{h}_{P1})}_{=0}\Big]\cdot d\mathbf{s}+\\&R_{2,1}\iint_{S_2}\Big[\underbrace{(-\mathbf{e}_{P2}\times\mathbf{h}_{P2}-\mathbf{e}_{P2}\times\mathbf{h}_{P2})}_{=-4}+\Gamma_2\underbrace{(\mathbf{e}_{P2}\times\mathbf{h}_{P2}-\mathbf{e}_{P2}\times\mathbf{h}_{P2})}_{=0}\Big]\cdot d\mathbf{s}=0.\end{aligned}\tag{10.15}$$

[1] It can be shown that a propagating mode in a transmission line or a waveguide can be represented by real-valued phasor vectors **e** and **h**.

Since the fields $(\mathbf{e}_{\mathrm{P}n}, \mathbf{h}_{\mathrm{P}n})$, $n=1,2$, correspond to 1 W of transferred power, their respective integrals over the port cross-sections (integration over the Poynting vector) have the same value:

$$\frac{1}{2}\iint_{S_n} (\mathbf{e}_{\mathrm{P}n} \times \mathbf{h}_{\mathrm{P}n}) \cdot d\mathbf{s} = 1,\ n = 1,2. \tag{10.16}$$

It follows from (10.15) and (10.16) that

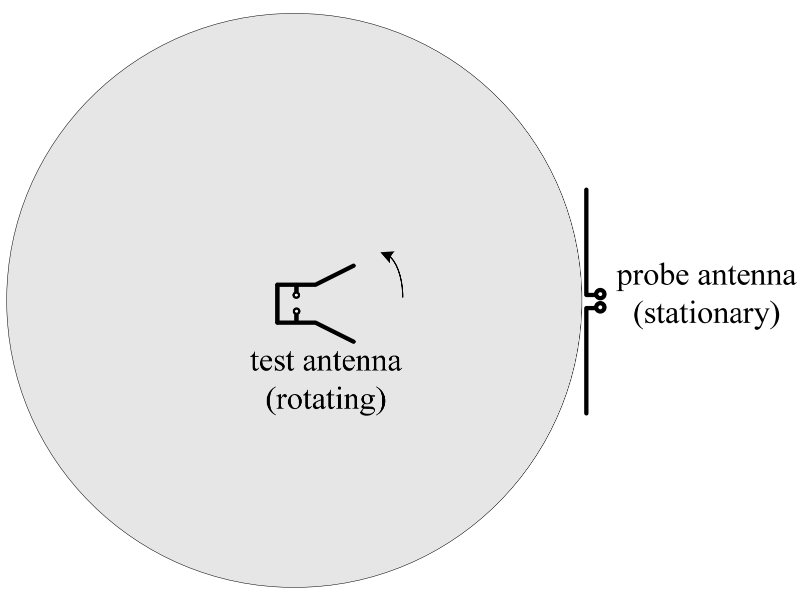
$$R_{1,2} = R_{2,1}. \tag{10.17}$$

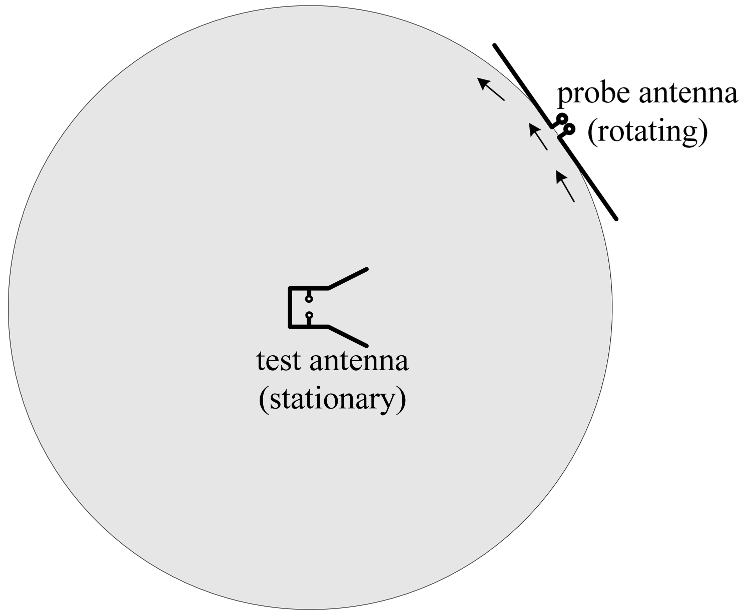
This result together with (10.13) and (10.14) leads to the conclusion that the <u>received-to-transmitted power ratio in a two-antenna system does not depend on which antenna transmits and which receives</u>.

1.4. <u>Reciprocity of the radiation pattern</u>

The measured radiation pattern of an antenna is the same in receiving and in transmitting mode if the system is linear. Nonlinear devices such as diodes and transistors make the system nonlinear, therefore, nonreciprocal.

In a two-antenna pattern measurement system, the pattern would not depend on whether the antenna under test (AUT) receives and the other antenna transmits, or *vice versa*. It also does not matter whether the AUT rotates and the other antenna is stationary, or *vice versa*. What matters is the mutual orientation of the antennas.

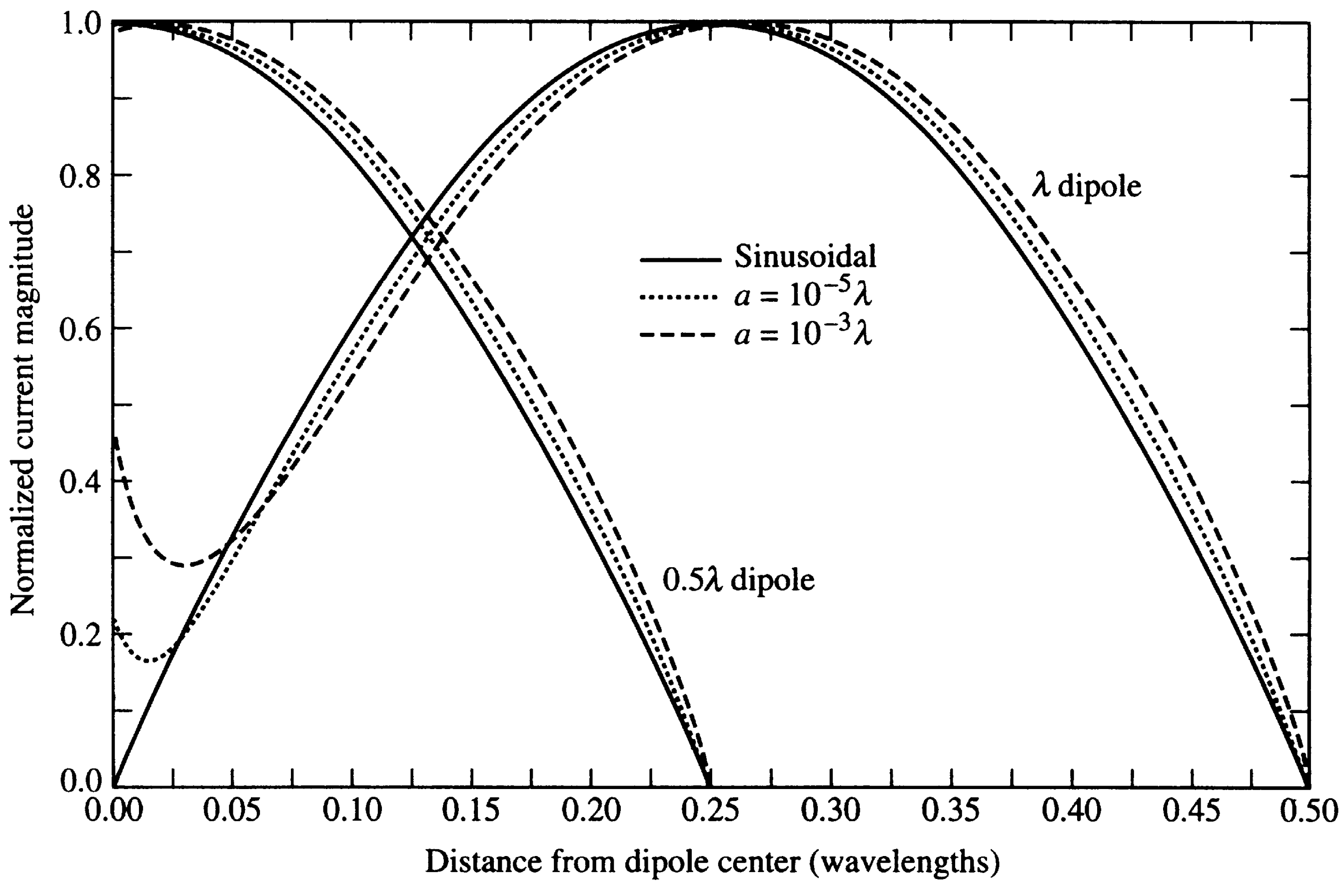



probe antenna
(rotating)
test antenna
(stationary)

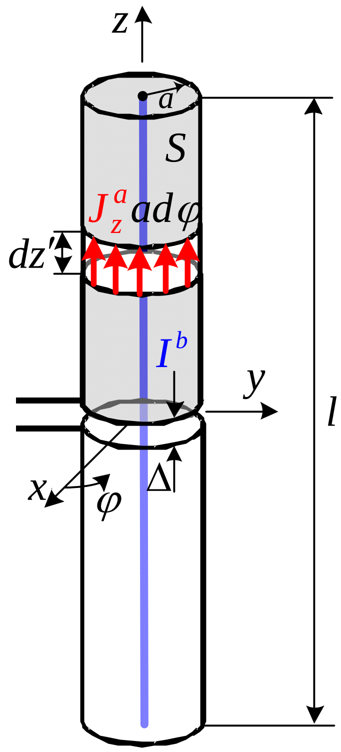
2. Self-impedance of a dipole using the induced *emf* method

The induced ***emf*** (electro-motive force) method was developed by Carter[2] in 1932, when computers were not available and analytical (closed-form) solutions were much needed to calculate the self-impedance of wire antennas. The method was later extended to calculate mutual impedances of multiple wires (see, e.g., Elliot, *Antenna Theory and Design*). The ***emf*** method is restricted to straight parallel wires.

Measurements and full-wave simulations indicate that the current distribution on thin dipoles is nearly sinusoidal (except at the current minima). The induced ***emf*** method assumes this type of idealized distribution. It results in satisfactory accuracy for dipoles with length-diameter ratios as small as 100.



[2] P.S. Carter, "Circuit relations in radiating systems and applications to antenna problems," *Proc. IRE*, **20**, pp.1004-1041, June 1932.



Consider a tubular dipole the arms of which are made of perfect electric conductor (PEC). When excited by a voltage-gap source at its base, the dipole supports surface current along z, which radiates. This surface current density is $J_{sz}(z') = H_{\varphi}(z')$ as per the boundary conditions at the PEC surface where $E_z(z') = 0$.

Using the equivalence principle, we consider an equivalent problem where $J_{sz}^a(z') = J_{sz}(z')$ is a cylindrical current sheet that exists over a closed cylindrical surface S tightly enveloping the dipole. It radiates in open space generating the field $(\mathbf{E}^a, \mathbf{H}^a)$ where

$$E_z^a(\rho \le a, z') = \begin{cases} V_{in} / \Delta, & -\Delta/2 \le z' \le \Delta/2, \\ 0, & \text{elsewhere.} \end{cases} \quad (10.18)$$

Here, Δ is the feed gap length.

Consider also a linear current source $I^b(z')$ along the axis of the cylinder (the z axis) where I^b is nonzero only for $-l/2 < z' < l/2$. It also radiates in open space and its field is denoted as $(\mathbf{E}^b, \mathbf{H}^b)$.

In the volume bound by S, we apply the reciprocity formula (10.7):

$$\iiint\limits_{V_{[S]}} (\mathbf{E}^a \cdot \mathbf{J}^b - \mathbf{E}^b \cdot \mathbf{J}^a) dv = 0. \quad (10.19)$$

Bearing in mind the surface nature of source a, the linear nature of source b, and equation (10.18), we write (10.19) as

$$\int_0^{2\pi} \int_{-l/2}^{l/2} E_z^b J_{sz}^a a dz' d\varphi = \int_{-\Delta/2}^{\Delta/2} E_z^a I^b dz'. \quad (10.20)$$

In (10.20), we have neglected the edge effects from the disc-like end surfaces of S since a (the dipole radius) is at least 100 times smaller than its length l. We also assumed that the electric fields of the two sources have only z components.

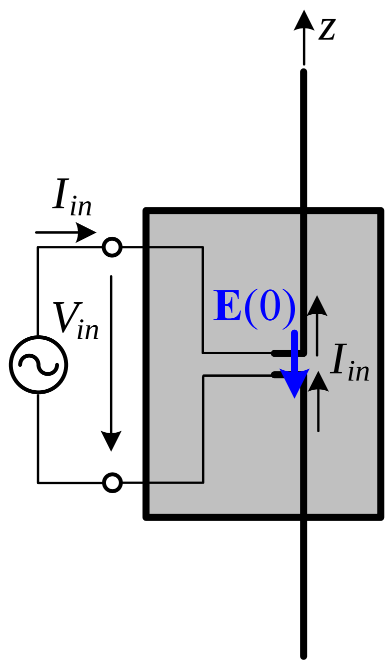
Assuming constant current distribution in the feed gap, we obtain

$$\int_0^{2\pi} \int_{-l/2}^{l/2} E_z^b J_{sz}^a a dz' d\varphi = -I_{in} V_{in} \quad (10.21)$$

where

$$V_{in} = -\int_{-\Delta/2}^{\Delta/2} E_z^a dz' \tag{10.22}$$

is the voltage on the terminals of the generator driving the current I^b. The minus sign in (10.22) reflects the fact that a positive V_{in}, which implies a "positive" current I_{in}, i.e., current flowing in the positive z direction, relates to a "negative" electric field at the dipole's base, i.e., E_z points in the negative z direction. This is illustrated below where the dipole antenna is viewed as a 1-port network.



Further, due to the cylindrical symmetry, all quantities in the integral in (10.21) are independent of φ. Thus,

$$\int_{-l/2}^{l/2} E_z^b \left(2\pi a J_{sz}^a\right) dz' = -I_{in} V_{in} . \tag{10.23}$$

The quantity in the brackets in (10.23) is the total current I^a at position z':

$$\int_{-l/2}^{l/2} E_z^b I^a dz' = -I_{in} V_{in} . \tag{10.24}$$

We now require that I^a represents the actual current distribution along the dipole's arms and we drop the superscripts:

$$\int_{-l/2}^{l/2} E_z I dz' = -I_{in} V_{in} . \tag{10.25}$$

Note that E_z is the field at the cylindrical surface enveloping the dipole due to I^b, the distribution of which along z' is also representing the actual current distribution (i.e., it is identical to I^a). The above result leads to the following self-impedance expression:

$$Z_{in}\Big|_{z'=0} = \frac{V_{in}}{I_{in}} = \frac{V_{in} \cdot I_{in}}{I_{in}^2} = -\frac{1}{I_{in}^2} \int_{-l/2}^{l/2} E_z(z') \cdot I(z') dz' . \tag{10.26}$$

In the classical ***emf*** method, we assume that the current has a sinusoidal distribution:

$$I(z') = \begin{cases} I_0 \sin\left[\beta\left(\dfrac{l}{2} - z'\right)\right], & 0 \le z' \le l/2 \\ I_0 \sin\left[\beta\left(\dfrac{l}{2} + z'\right)\right], & -l/2 \le z' \le 0 \end{cases} \tag{10.27}$$

So far, we have obtained only the far-field components of the field generated by the current in (10.27) (see Lecture 9). However, when the input resistance and reactance are needed, the near field must be known. In our case, we are interested in E_z, which is the field produced by $I(z')$ as if there is no conductor surface present. If we know it, we can calculate the integral in (10.26) since we already know $I(z')$ from (10.27).

We use cylindrical coordinates to describe the locations of the integration point (primed coordinates) and the observation point. The electric field can be expressed in terms of the VP **A** and the scalar potential ϕ (see Lecture 2):

$$\mathbf{E} = -\nabla\phi - j\omega\mathbf{A}, \tag{10.28}$$

$$\Rightarrow E_z = -\frac{\partial\phi}{\partial z} - j\omega A_z . \tag{10.29}$$

The VP **A** is *z*-polarized,

$$A_z = \frac{\mu}{4\pi}\int_{-l/2}^{l/2} I_z(z')\frac{e^{-j\beta R}}{R}dz'. \tag{10.30}$$

The scalar potential is

$$\phi = \frac{1}{4\pi\varepsilon}\int_{-l/2}^{l/2} q_l(z')\frac{e^{-j\beta R}}{R}dz'. \tag{10.31}$$

Here, q_l stands for linear charge density (C/m). Knowing that the current depends only on z', the continuity relation is written as

$$j\omega q_l = -\frac{\partial I_z}{\partial z'}. \tag{10.32}$$

$$\Rightarrow q_l(z') = \begin{cases} -j\dfrac{I_0}{c}\cos\left[\beta\left(\dfrac{l}{2}-z'\right)\right], & 0 \le z' \le l/2 \\ +j\dfrac{I_0}{c}\cos\left[\beta\left(\dfrac{l}{2}+z'\right)\right], & -l/2 \le z' \le 0 \end{cases} \tag{10.33}$$

where $c = \omega/\beta$ is the speed of light. Now we express **A** and ϕ as

$$A_z = \frac{\mu}{4\pi}I_0\left\{\int_{-l/2}^{0}\sin\left[\beta\left(\frac{l}{2}+z'\right)\right]\frac{e^{-j\beta R}}{R}dz' + \int_{0}^{l/2}\sin\left[\beta\left(\frac{l}{2}-z'\right)\right]\frac{e^{-j\beta R}}{R}dz'\right\} \tag{10.34}$$

$$\phi = j\frac{\eta I_0}{4\pi}\left\{-\int_{-l/2}^{0}\cos\left[\beta\left(\frac{l}{2}+z'\right)\right]\frac{e^{-j\beta R}}{R}dz' + \int_{0}^{l/2}\cos\left[\beta\left(\frac{l}{2}-z'\right)\right]\frac{e^{-j\beta R}}{R}dz'\right\}. \tag{10.35}$$

Here, $\eta = \sqrt{\mu/\varepsilon}$ is the intrinsic impedance of the medium.

The distance between integration and observation point is

$$R = \sqrt{\rho^2 + (z-z')^2}. \tag{10.36}$$

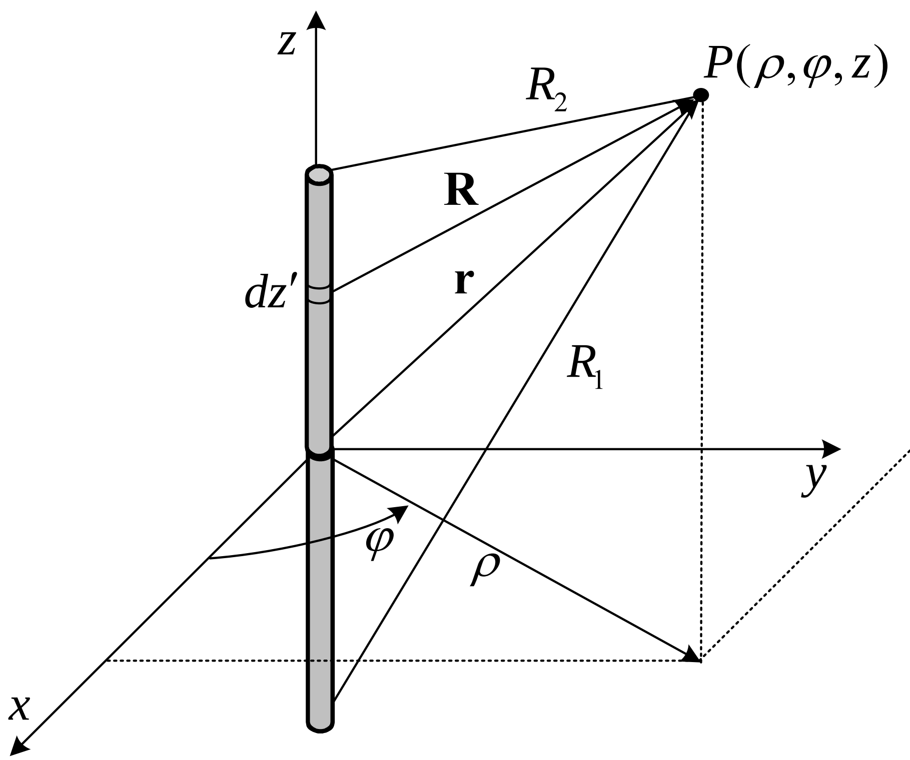
Equation (10.36) is substituted in (10.34) and (10.35). In addition, the resulting equations for A_z and ϕ are modified making use of Moivre's formulas:

$$\begin{aligned} \cos x &= \frac{1}{2}\left(e^{jx}+e^{-jx}\right) \\ \sin x &= \frac{1}{2j}\left(e^{jx}-e^{-jx}\right) \end{aligned}, \text{ where } x = \beta\left(\frac{l}{2}\pm z'\right). \tag{10.37}$$

Then, the equations for A_z and ϕ are substituted in (10.29) to derive the expression for E_z valid at any observation point. This is a rather lengthy derivation, and we give the final result only:

$$E_z = -j\frac{\eta I_0}{4\pi}\left[\frac{e^{-j\beta R_1}}{R_1} + \frac{e^{-j\beta R_2}}{R_2} - 2\cos\left(\frac{\beta l}{2}\right)\frac{e^{-j\beta r}}{r}\right]. \qquad (10.38)$$

Here, r is the distance from the observation point to the dipole's center, while R_1 and R_2 are the distances to the lower and upper edges of the dipole, respectively (see figure below).



We need $E_z(z')$ at the dipole's surface where

$$r \approx z', \; R_1 = z' + l/2, \text{ and } R_2 = (l/2) - z'. \qquad (10.39)$$

Notice the thin-wire approximation! The final goal of this development is to find the self-impedance (10.26) of the dipole. We substitute (10.39) in (10.38). The result for $E_z(z')$ is then substituted in (10.26), and the integration is performed. We give the final results for the real and imaginary parts of Z_{in}:

$$R_{in} = k\frac{\eta}{2\pi}\cdot\left\{C + \ln(\beta l) - C_i(\beta l) + \frac{1}{2}\sin(\beta l)\left[S_i(2\beta l) - 2S_i(\beta l)\right] + \right.$$

$$\left. + \frac{1}{2}\cos(\beta l)\left[C + \ln\left(\frac{\beta l}{2}\right) + C_i(2\beta l) - 2C_i(\beta l)\right]\right\} = k\frac{\eta}{2\pi}\cdot\Im\ , \tag{10.40}$$

$$X_{in} = k\frac{\eta}{4\pi}\left\{2S_i(\beta l) - \cos(\beta l)\left[S_i(2\beta l) - 2S_i(\beta l)\right] + \right.$$

$$\left. + \sin(\beta l)\left[C_i(2\beta l) - 2C_i(\beta l) + C_i\left\{2\beta a^2 / l\right\}\right]\right\}\ , \tag{10.41}$$

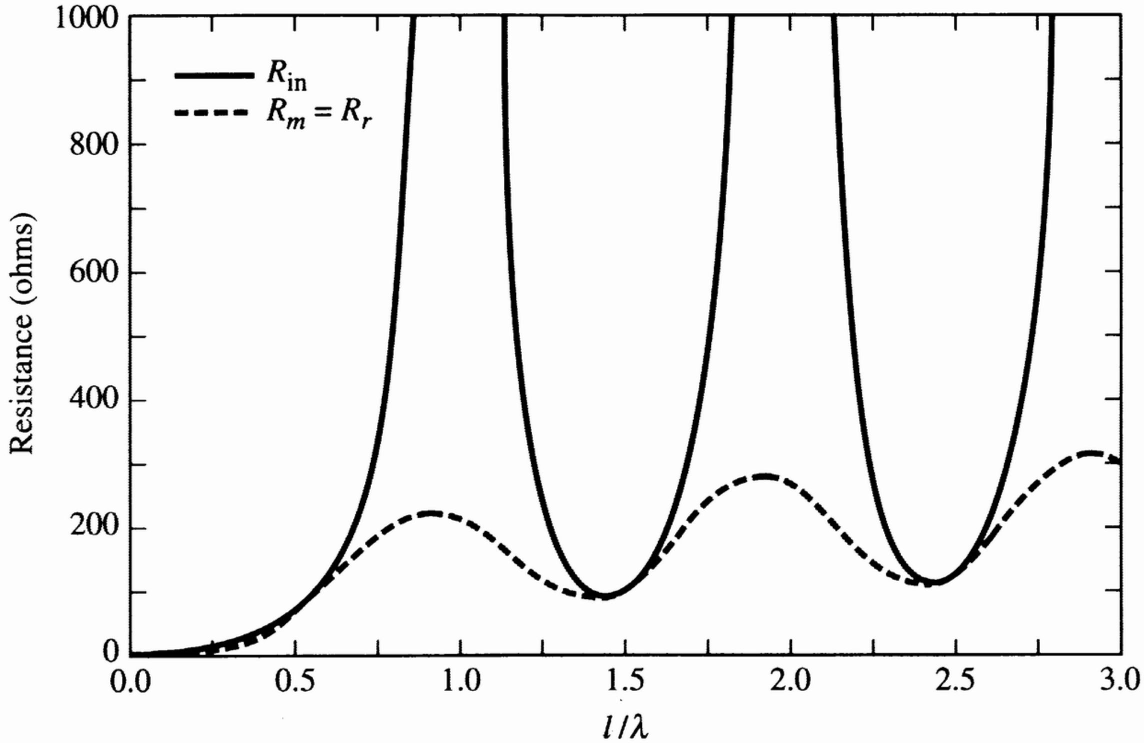
where $k = 1/\sin^2(\beta l/2)$ is the coefficient accounting for the difference between the maximum current magnitude along the dipole and the magnitude of the input current at the dipole's center [see Lecture 9, section 2].

Equation (10.40) is identical with the expression found for the input resistance of an infinitesimally thin wire [see Lecture 9, Eqs. (9.37) and (9.38)]. Expression (10.41) for the dipole's reactance however is new. For a small dipole, the input reactance can be approximated by

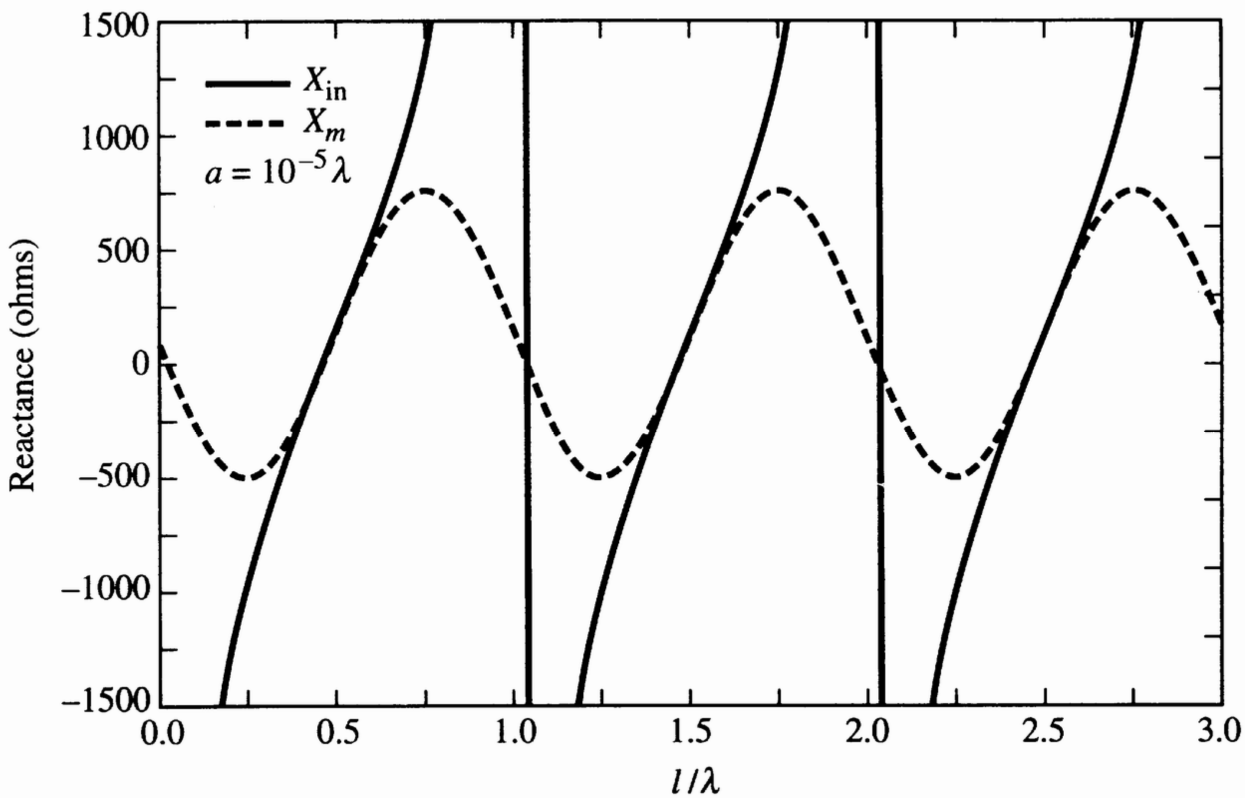
$$X_{in} \approx -120\frac{\left[\ln(l/a) - 1\right]}{\tan(\beta l)}. \tag{10.42}$$

The results produced by (10.40) and (10.41) for different ratios l/λ are given in the plots below.

INPUT IMPEDANCE OF A THIN DIPOLE (***EMF*** METHOD) OF RADIUS $a = 10^{-5}\lambda$

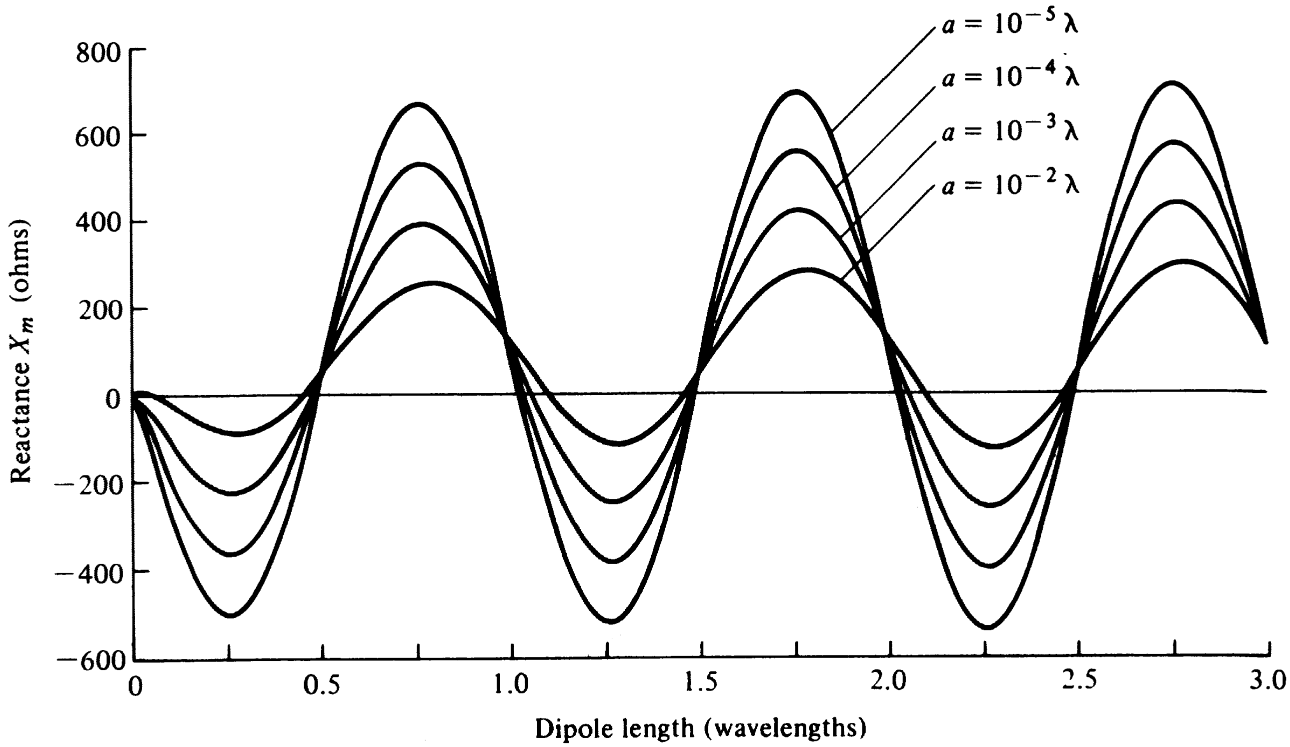


(a) Resistance



(b) Reactance

INPUT REACTANCE OF A THIN DIPOLE (***EMF*** METHOD) FOR DIFFERENT RADII a



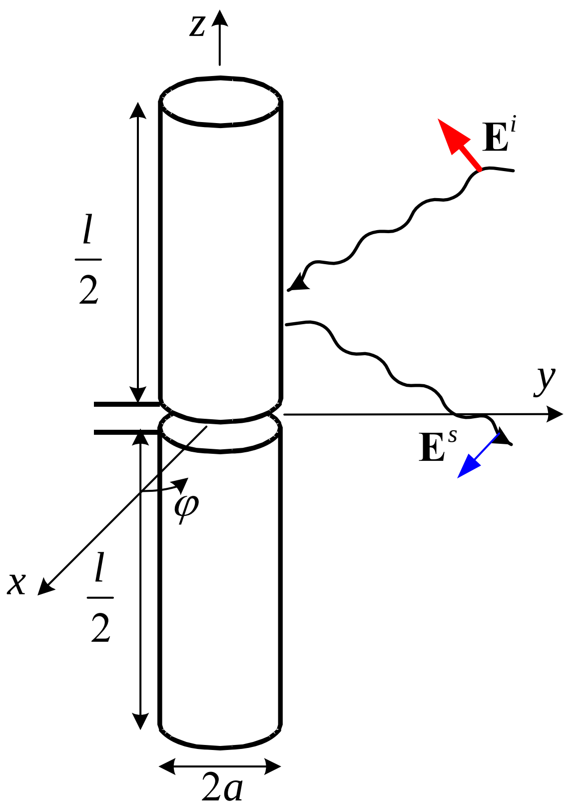
Note that:

- the reactance does not depend on the radius a, when the dipole length is a multiple of a half-wavelength ($l = n\lambda / 2$), as follows from (10.41);
- the resistance does not depend on a according to the assumptions made in the ***emf*** method (see equation (10.40)).

4. Pocklington's equation

The assumption of a sinusoidal current distribution along the dipole is considered accurate enough for wire diameters $d < 0.05\lambda$. The current distribution is not quite sinusoidal in the case of thicker wires. The currents must be computed using some general numerical approach. Below, we introduce two integral equations, which can produce the current distribution on any straight wire antenna of finite diameter. These equations are classical in wire antenna theory. We do not discuss in detail their numerical solutions since they are out of the scope of this course.

To derive Pocklington's equation, the concepts of incident and scattered field are introduced first.



The *incident* wave is a wave produced by some external sources. This wave would have existed in the location of the scatterer, if the scatterer were not present. The scatterer however is present, and, since it is a conducting body, it requires vanishing electric field components tangential to its surface,

$$\mathbf{E}_\tau^t = 0 . \tag{10.43}$$

The vector $\mathbf{E}^t$ denotes the so-called total electric field. This means that as the non-zero incident field impinges upon the conducting scatterer, it induces on its surface currents $\mathbf{J}_s$, which in their turn produce a field, the *scattered* field $\mathbf{E}^s$. The scattered and the incident fields superimpose to produce the total field

$$\mathbf{E}^t = \mathbf{E}^i + \mathbf{E}^s . \tag{10.44}$$

The scattered field is such that (10.43) is fulfilled, i.e.,

$$\mathbf{E}_\tau^s = -\mathbf{E}_\tau^i . \tag{10.45}$$

Any object presenting a discontinuity in the wave's path is a scatterer, and so is any receiving antenna.

The above concepts hold for transmitting antennas, too. In the case of a wire dipole, the incident field exists only at the base of the dipole (in its feed gap).

In the case of a cylindrical dipole with excitation of cylindrical symmetry, the $\mathbf{E}$ field has no φ-component and is independent of the φ coordinate. The only tangential component is E_z. The boundary condition at the dipole's surface is

$$E_z^s = -E_{z/\rho=a,-l/2\le z\le l/2}^i . \tag{10.46}$$

The scattered field can be expressed in terms of $\mathbf{A}$ and ϕ as it was already done in (10.29):

$$E_z^s = -\frac{\partial \phi}{\partial z} - j\omega A_z = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial z^2} - j\omega A_z , \tag{10.47}$$

or

$$E_z^s = -j\frac{1}{\omega\mu\varepsilon}\left(\beta^2 A_z + \frac{\partial^2 A_z}{\partial z^2}\right). \tag{10.48}$$

We assume only z-components of the surface currents and no edge effects:

$$A_z(\rho,\varphi,z) = \frac{\mu}{4\pi}\int_{-l/2}^{l/2}\int_0^{2\pi} J_{sz}\frac{e^{-j\beta R}}{R}\underbrace{a\,d\varphi' dz'}_{ds}. \tag{10.49}$$

If the cylindrical symmetry of the dipole and the excitation are preserved, the current J_z does not depend on the azimuthal angle φ. It can be shown that the field created by a cylindrical sheet of surface currents J_{sz} is equivalent to the field created by a current filament of current I_z,

$$2\pi a J_{sz} = I_z \Rightarrow J_{sz}(z') = \frac{1}{2\pi a}I_z(z'). \tag{10.50}$$

Then, (10.49) reduces to

$$A_z(\rho,\varphi,z) = \frac{\mu}{4\pi}\int_{-l/2}^{l/2}\frac{1}{2\pi a}\int_0^{2\pi} I_z(z')\frac{e^{-j\beta R}}{R}a\,d\varphi' dz'. \tag{10.51}$$

The distance between observation and integration points is

$$\begin{aligned} R &= \sqrt{(x-x')^2+(y-y')^2+(z-z')^2} = \\ &= \sqrt{\rho^2+a^2-2\rho a\cos(\varphi-\varphi')+(z-z')^2}\ . \end{aligned} \tag{10.52}$$

The cylindrical geometry of the problem implies the cylindrical symmetry of the observed fields, i.e., **A** does not depend on φ. We assume that $\varphi = 0$. Besides, we are interested in the scattered field produced by this equivalent current at the dipole's surface, i.e., the observation point is at $\rho = a$. Then,

$$A_z(a,0,z) = \mu\int_{-l/2}^{l/2} I_z(z')\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{-j\beta R}}{4\pi R}d\varphi'\right)dz' = \mu\int_{-l/2}^{l/2} I_z(z')G\left(z,z'\right)dz', \tag{10.53}$$

where

$$G\left(z,z'\right) = \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{-j\beta R}}{4\pi R}d\varphi', \tag{10.54}$$

$$R_{(\rho=a,\varphi=0)} = \sqrt{4a^2 \sin^2\left(\frac{\varphi'}{2}\right) + (z-z')^2} \, . \tag{10.55}$$

Substituting (10.53) in (10.48) yields

$$E_z^s(\rho=a) = -j\frac{1}{\omega\varepsilon}\left(\beta^2 + \frac{d^2}{dz^2}\right)\int_{-l/2}^{l/2} I_z(z')G(z,z')dz' \, . \tag{10.56}$$

Imposing the boundary condition (10.46) on the field in (10.56) leads to

$$\left(\beta^2 + \frac{d^2}{dz^2}\right)\int_{-l/2}^{l/2} I_z(z')G(z,z')dz' = -j\omega\varepsilon E_z^i(\rho=a) \, . \tag{10.57}$$

The source I_z does not depend on z and (10.57) can be rewritten as

$$\int_{-l/2}^{l/2} I_z(z')\left(\beta^2 G(z,z') + \frac{d^2 G(z,z')}{dz^2}\right)dz' = -j\omega\varepsilon E_z^i(\rho=a) \, . \tag{10.58}$$

Equation (10.58) is called Pocklington's[3] integro-differential equation. It is used to compute the equivalent filamentary current distribution $I_z(z')$ by knowing the incident field on the dipole's surface.

When the gap of length b is the only place where E_z^i exists, equation (10.58) is written as

$$\int_{-l/2}^{l/2} I_z(z')\left(\beta^2 G(z,z') + \frac{d^2 G(z,z')}{dz^2}\right)dz' = \begin{cases} -j\omega\varepsilon E_z^i, & -\frac{b}{2} \le z \le \frac{b}{2} \\ 0, & \frac{b}{2} < |z| < \frac{l}{2} \end{cases} \tag{10.59}$$

If we assume that the wire is very thin, then the Green's function $G(z,z')$ (10.54) simplifies to

$$G(z,z') = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-j\beta R}}{4\pi R} d\varphi' = \frac{e^{-j\beta R}}{4\pi R}, \tag{10.60}$$

since R reduces to $R_{(\rho=a\to 0)} = \sqrt{4a^2 \sin^2(\varphi'/2) + (z-z')^2} \approx z - z'$. Assuming (10.60), Richmond[4] has differentiated and re-arranged (10.58) in a form more convenient for programming:

[3] H.C. Pocklington, "Electrical oscillation in wires", *Camb. Phil. Soc. Proc.*, **9**, 1897, pp.324-332.

$$\int_{-l/2}^{+l/2} I_z(z') \frac{e^{-j\beta R}}{4\pi R^5} \left[(1 + j\beta R)(2R^2 - 3a^2) + (\beta a R)^2 \right] dz' = -j\omega\varepsilon E_z^i . \quad (10.61)$$

Equation (10.61) can be solved numerically by the Method of Moments (MoM), after the structure is discretized into small linear segments.

4. Hallén's equation

Hallén's equation[5] can be derived as a modification of Pocklington's equation. It is easier to solve numerically, but it makes some additional assumptions. Consider again equation (10.59). It can be written in terms of A_z explicitly as (see also (10.48)):

$$\frac{d^2 A_z}{dz^2} + \beta^2 A_z = \begin{cases} -j\omega\varepsilon\mu E_z^i, & -\frac{b}{2} \le z \le \frac{b}{2} \\ 0, & \frac{b}{2} < |z| < \frac{l}{2} \end{cases} \quad (10.62)$$

When $b \to 0$, we can express the incident field in the gap via the voltage applied to the gap:

$$V_g = \lim_{b \to 0} b E_z^i . \quad (10.63)$$

The $E_z^i(z)$ function is an impulse function of z, such that

$$E_z^i = V_g \delta(z) . \quad (10.64)$$

The excitation term in (10.62) collapses into a δ-function:

$$\frac{d^2 A_z}{dz^2} + \beta^2 A_z = -j\omega\varepsilon\mu V_g \delta(z) . \quad (10.65)$$

If $z \ne 0$,

$$\frac{d^2 A_z}{dz^2} + \beta^2 A_z = 0 . \quad (10.66)$$

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[4] J.H. Richmond, "Digital computer solutions of the rigorous equations for scattering problems," *Proc. IEEE*, **53**, pp.796-804, August 1965.

[5] E. Hallén, "Theoretical investigation into the transmitting and receiving qualities of antennae," *Nova Acta Regiae Soc. Sci. Upsaliensis*, Ser. IV, No. 4, 1938, pp. 1-44.

Because the current density on the cylinder is symmetrical with respect to z', i.e., $J_z(z') = J_z(-z')$, the potential A_z must also be symmetrical. Then, the general solution of the ODE in (10.66) along z ($x = y = 0$) has the form:

$$A_z(z) = B\cos\left(\beta z\right) + C\sin\left(\beta\,|\,z\,|\right). \tag{10.67}$$

From (10.65) it follows that

$$\left.\frac{dA_z}{dz}\right|_{0_-}^{0+} = -j\omega\mu\varepsilon V_g. \tag{10.68}$$

From (10.67) and (10.68), we calculate the constant C:

$$\left.\frac{dA_z}{dz}\right|_{0_-}^{0+} = C\beta\cos(0_+) - C\left(-\beta\right)\cos(0_-) = -j\omega\mu\varepsilon V_g,$$

$$\Rightarrow 2C\beta = -j\omega\mu\varepsilon V_g,$$

$$\Rightarrow C = -j\sqrt{\mu\varepsilon}\,\frac{V_g}{2} = -j\frac{\mu}{\eta}\frac{V_g}{2}. \tag{10.69}$$

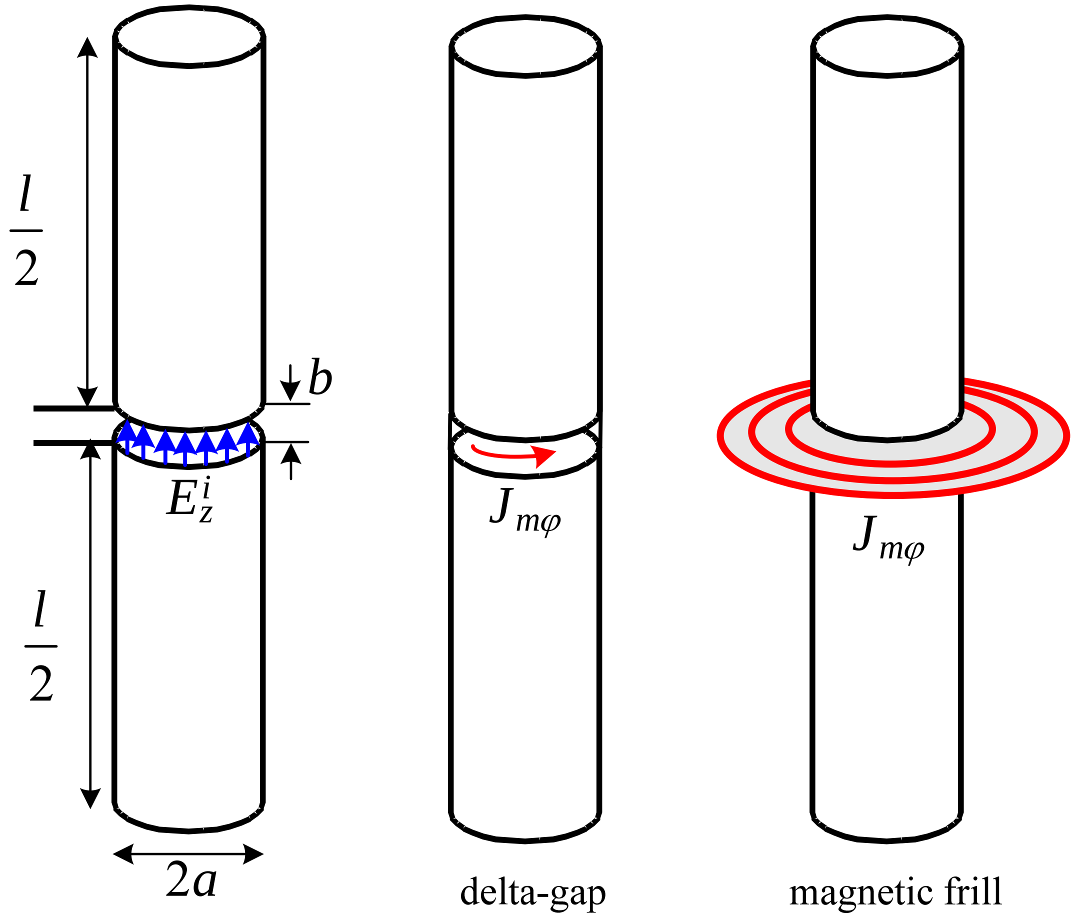
Equation (10.69) is substituted in (10.67), and A_z is expressed with its integral over the currents, to obtain the final form of Hallén's integral equation:

$$\int_{-l/2}^{+l/2} I_z(z')\frac{e^{-j\beta R}}{4\pi R}dz' = -j\frac{V_g}{2\eta}\sin\left(\beta\,|\,z\,|\right) + B\cos(\beta z). \tag{10.70}$$

Here, $R = \sqrt{a^2 + (z - z')^2}$. We must reiterate that *Hallén's equation assumes that the incident field exists only in the infinitesimal dipole gap*, while in Pocklington's equation there are no restrictions on the distribution of the incident field at the dipole.

5. Modeling the excitation field

- Delta-gap source (Pocklington and Hallén)
- Magnetic frill source (Pocklington)



$\frac{l}{2}$
b
E_z^i
$\frac{l}{2}$
$2a$
$J_{m\varphi}$
delta-gap
$J_{m\varphi}$
magnetic frill